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# Bandit Learning in Matching Markets

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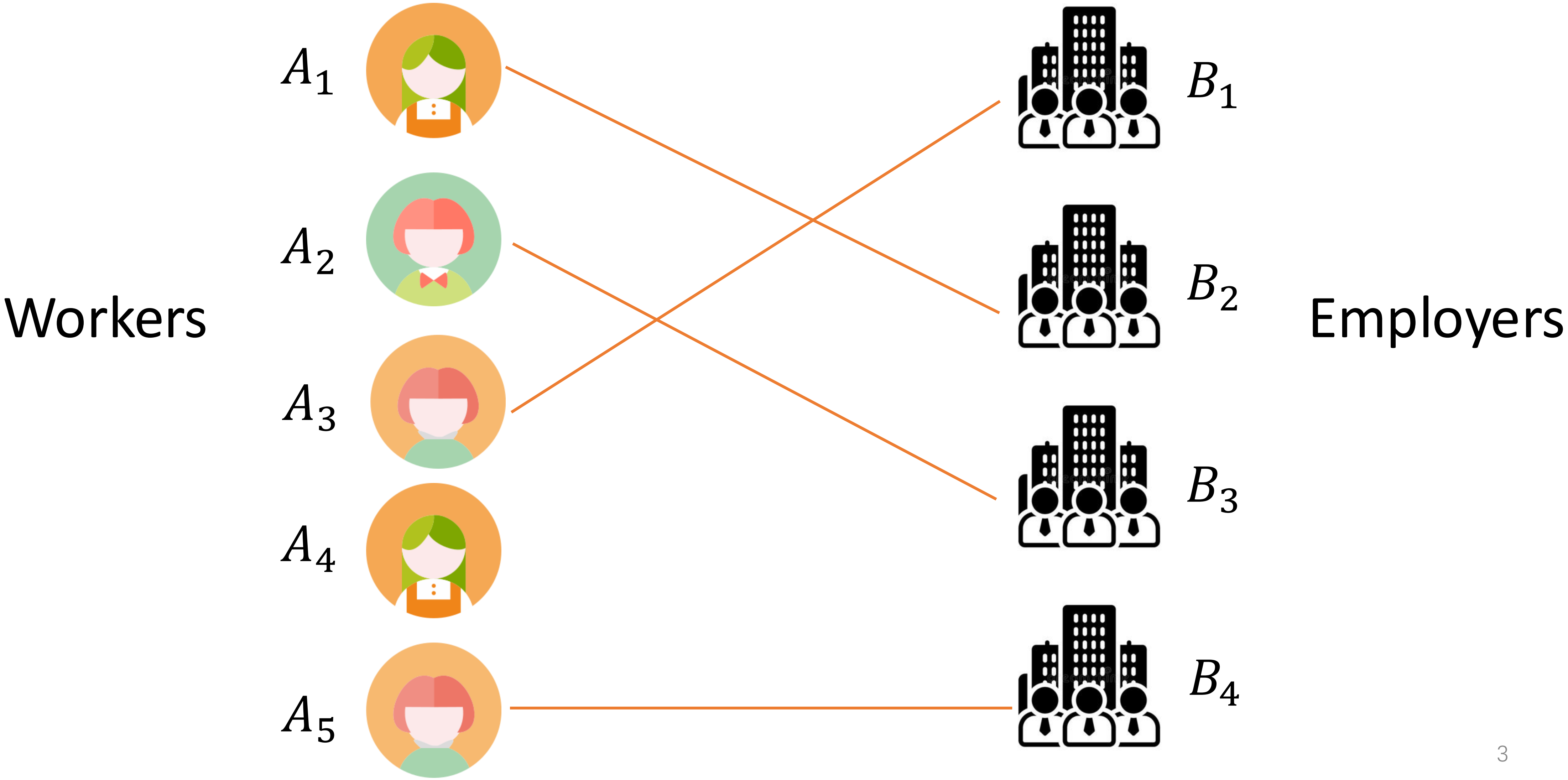
2025.4.13 at C&A

# Matching markets


- Talent cultivation (school admissions, student internships)
- Task allocation (crowdsourcing assignments, domestic services)
- Resource distribution (housing allocation, organ donation allocation)

https://www.freepik.com; https://twitter.com/IslingtonBC/status/1623340900725272578

# Matching market has two sides



# Both sides have preferences over the other side

Worker side

Based on payment or prior familiarity of the task

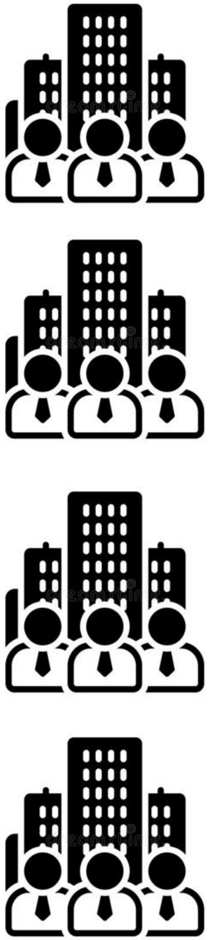
$A_1$ : $B_2 > B_3 > B_1 > B_4$

$A_2$ : $B_1 > B_2 > B_3 > B_4$

$A_3$ : $B_3 > B_1 > B_2 > B_4$

$A_4$ : $B_1 > B_2 > B_3 > B_4$

$A_5$ : $B_1 > B_2 > B_3 > B_4$



$B_1$

$B_2$

$B_3$

$B_4$

# Both sides have preferences over the other side

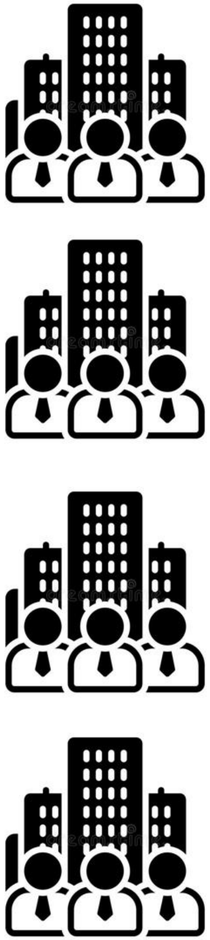
$A_1$

$A_2$

$A_3$

$A_4$

$A_5$



$B_1 : A_1 > A_2 > A_3 > A_4 > A_5$

$B_2 : A_2 > A_1 > A_4 > A_3 > A_5$

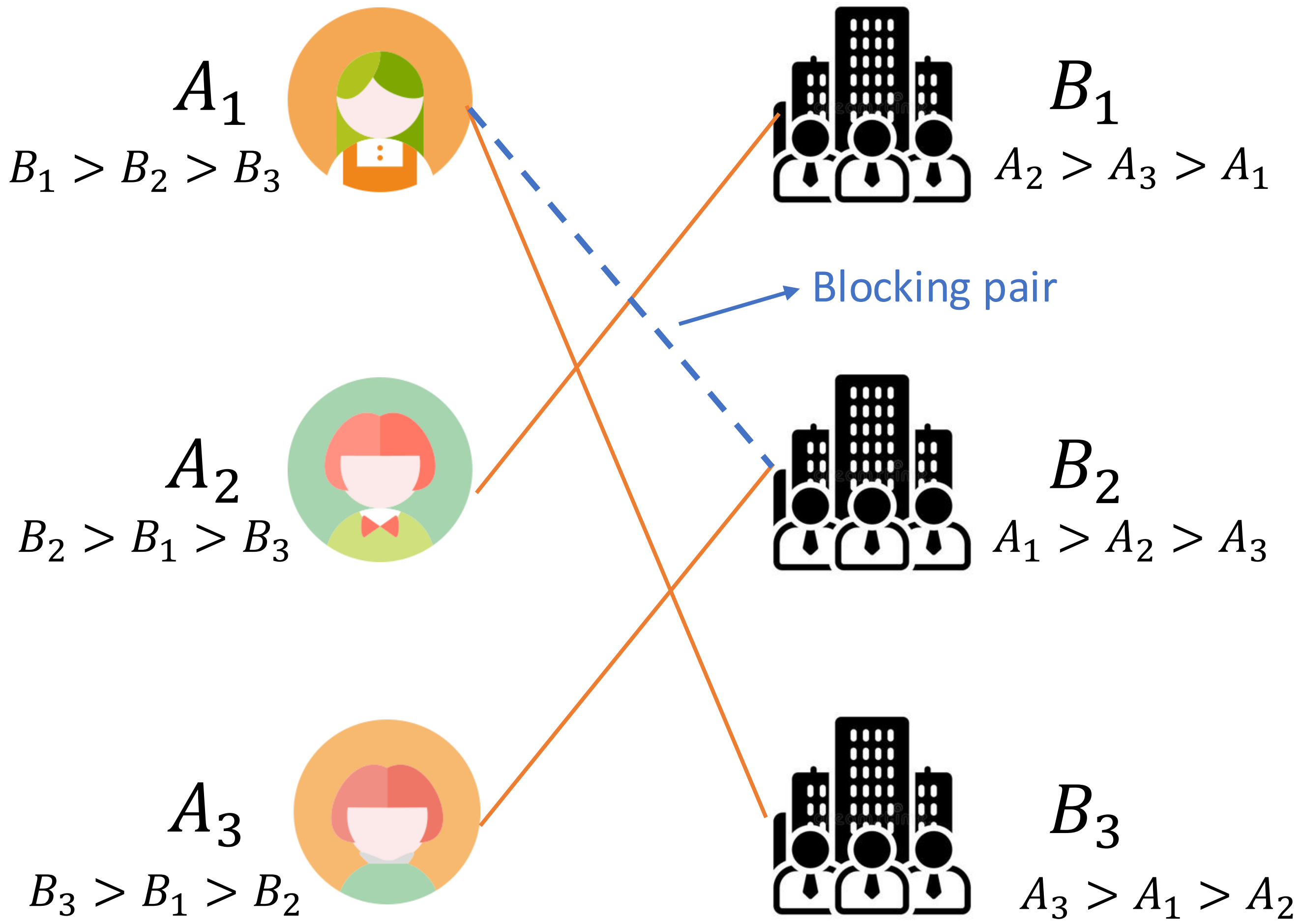
$B_3 : A_3 > A_1 > A_2 > A_5 > A_4$

$B_4 : A_4 > A_5 > A_1 > A_2 > A_3$

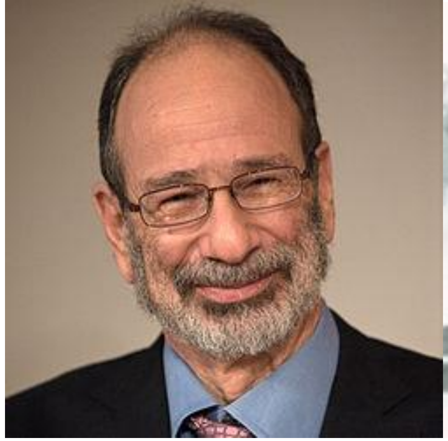
Employer side

Based on the skill levels of workers

# Stable matching




Nobel Memorial Prize
in Economic Sciences
2012



Alvin E. Roth

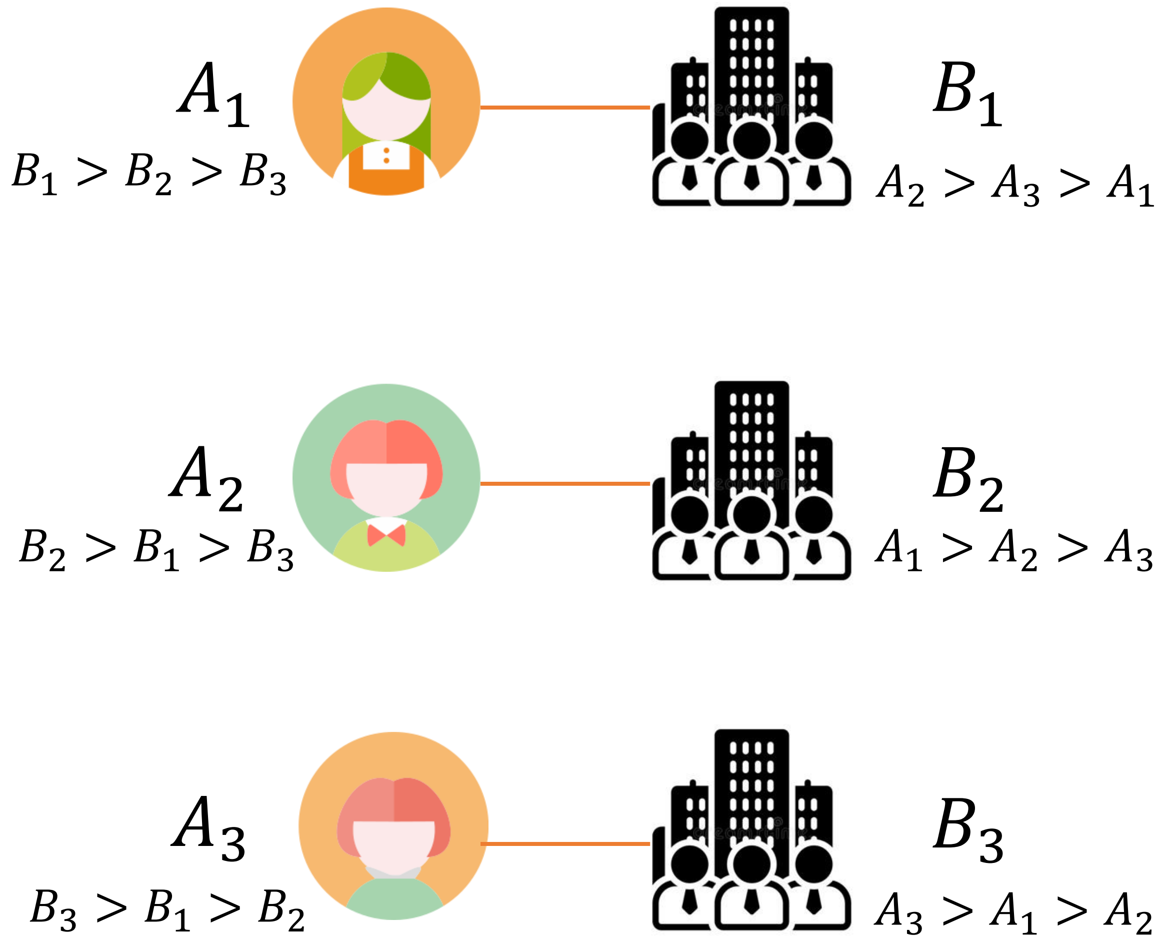
Lloyd Shapley

Participants have no incentive to abandon their current partner,
i.e.,
no blocking pair such that they both preferred to be matched with each other than their current partner

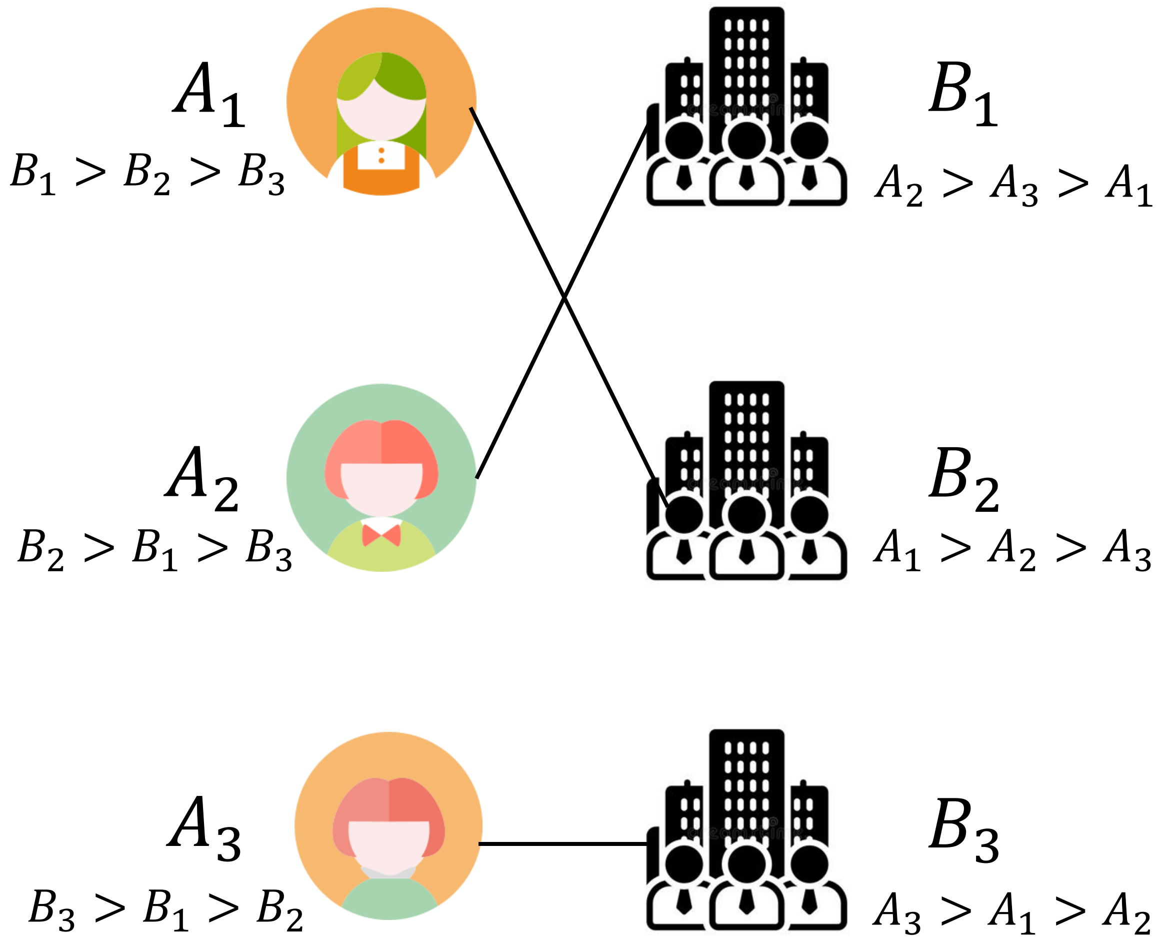


Alvin E. Roth and Lloyd S. Shapley jointly won the Nobel Prize in 2012 for their contributions to stable matching theory.

# May be more than one stable matchings

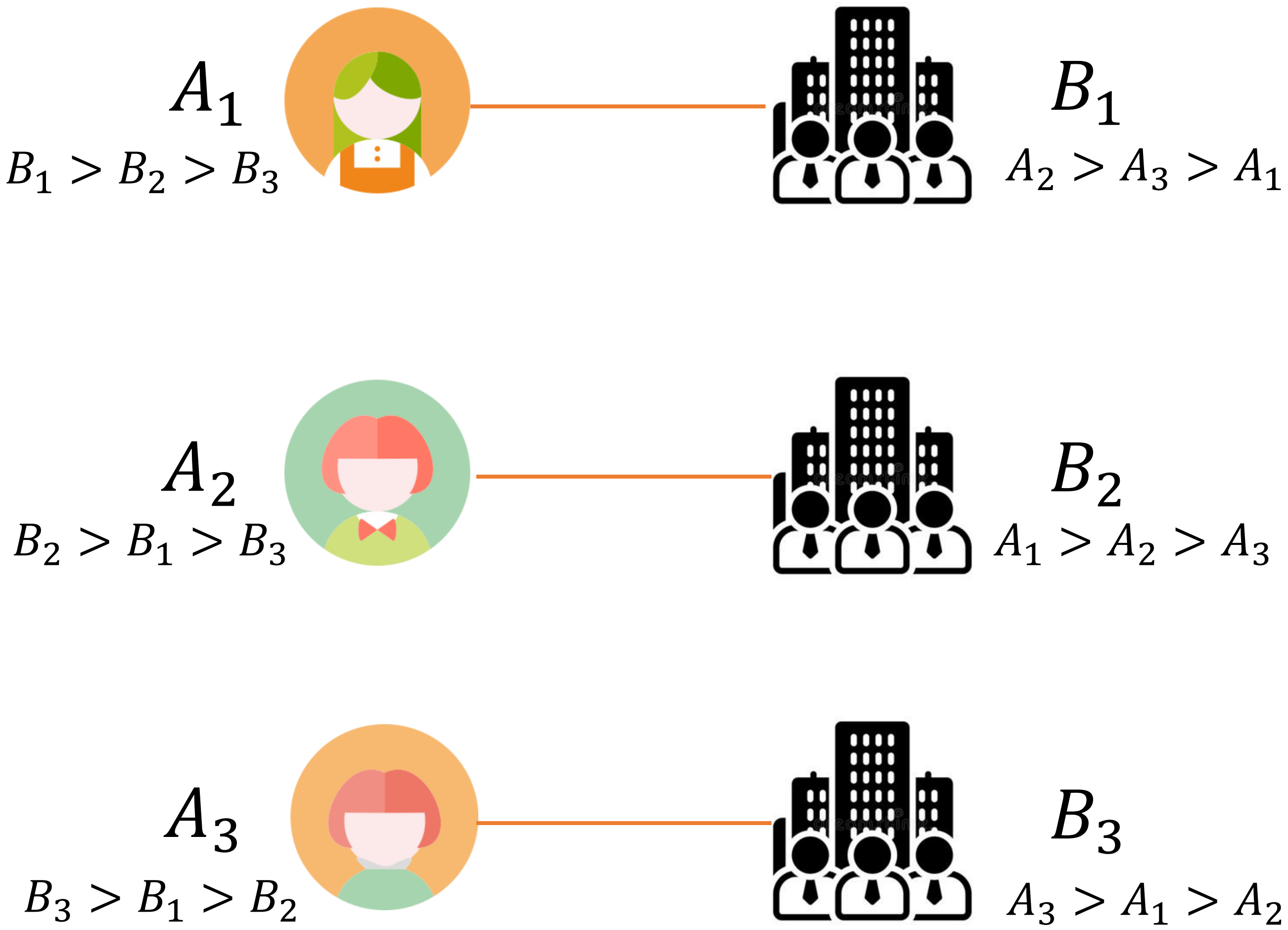



$$m_1 = \{(A_1, B_1), (A_2, B_2), (A_3, B_3)\}$$




$$m_2 = \{(A_1, B_2), (A_2, B_1), (A_3, B_3)\}$$

# A-side optimal stable matching[1]

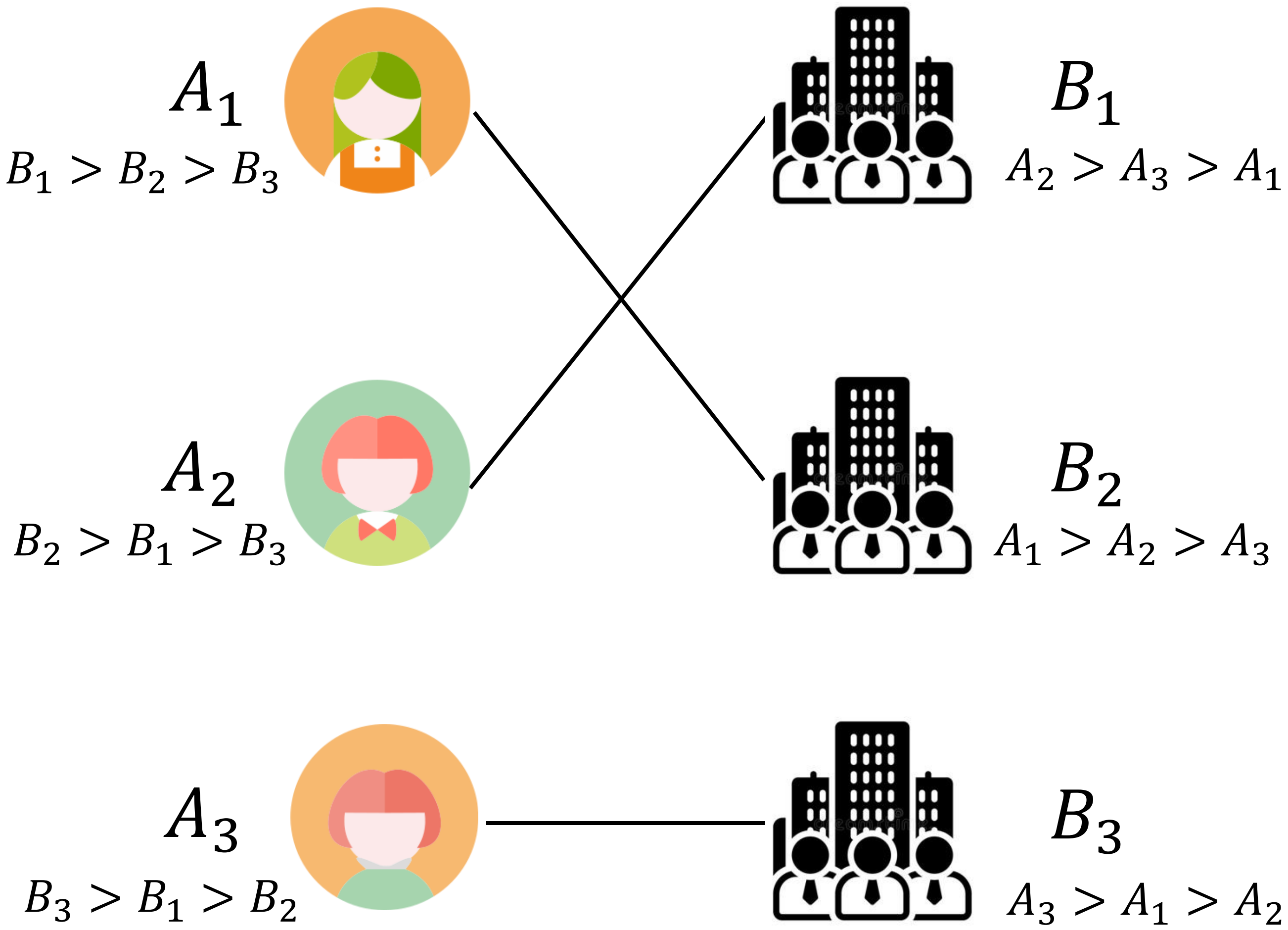



Each agent on A-side is matched with the most preferred partner among all stable matchings

$$m_1 = \{(A_1, B_1), (A_2, B_2), (A_3, B_3)\}$$

[1]The existence is proved by Gale and Shapley (1962).

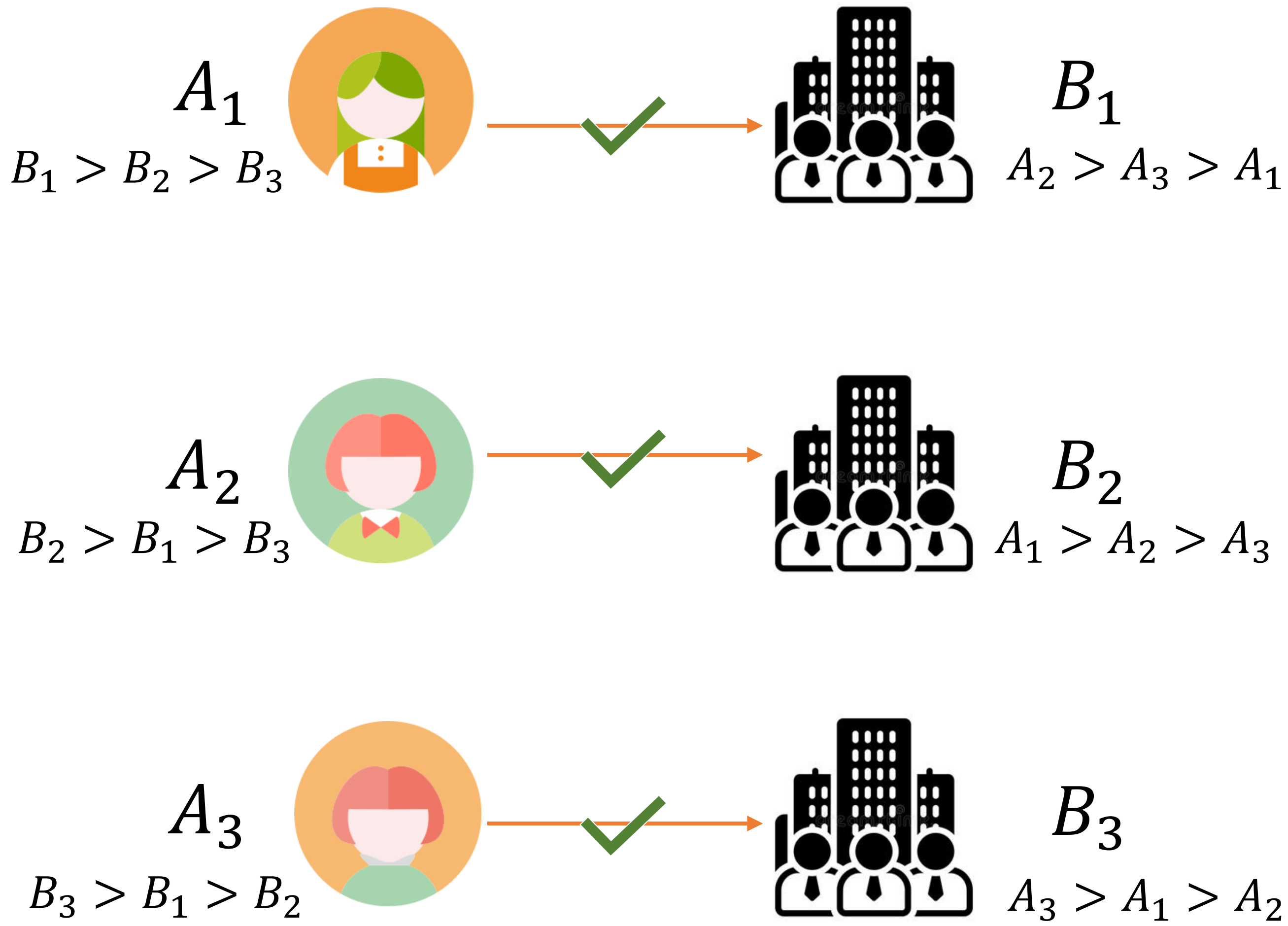
# A-side pessimal stable matching




Each agent on A-side is matched with the least preferred partner among all stable matchings

$$m_2 = \{(A_1, B_2), (A_2, B_1), (A_3, B_3)\}$$

# How to find a stable matching?

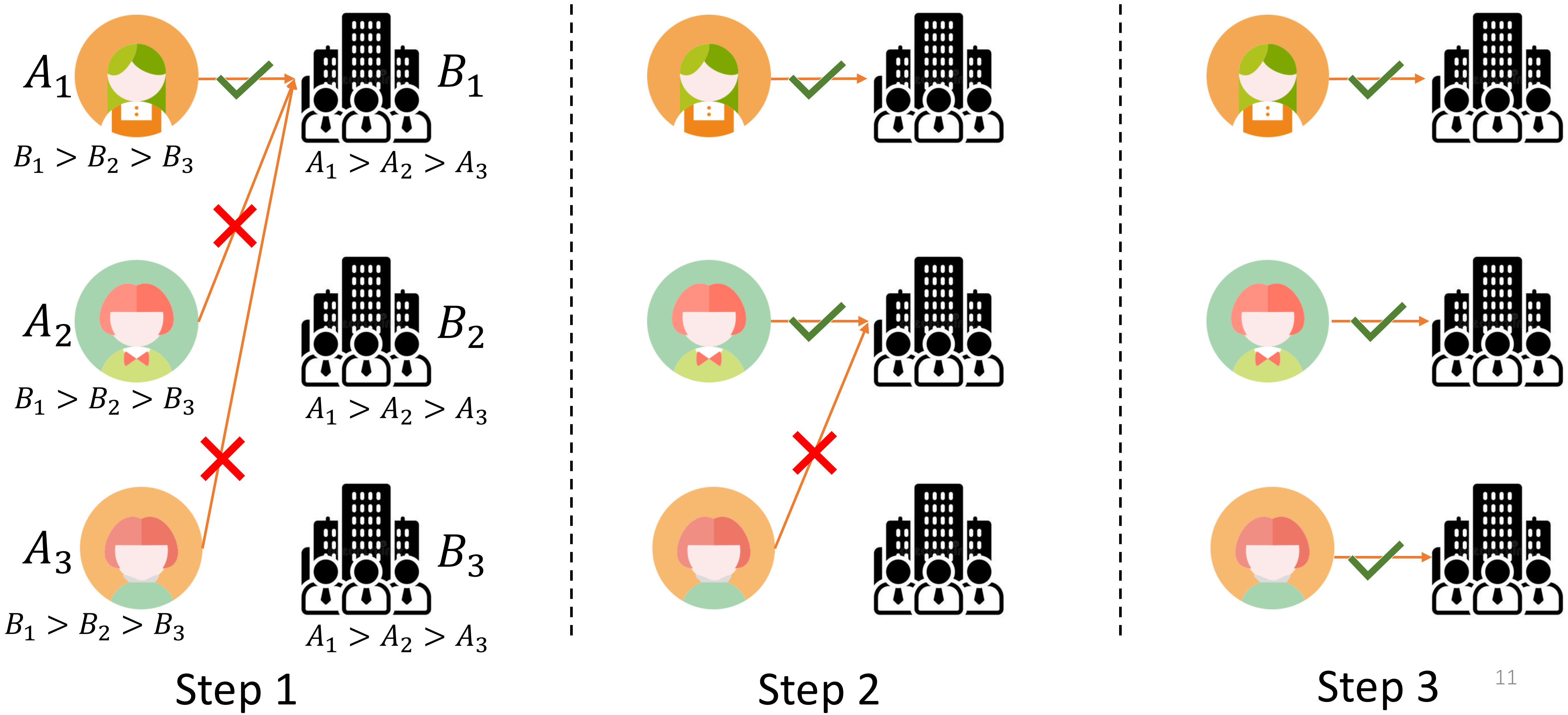



No rejection happens!

**Gale-Shapley (GS) algorithm**
[Gale and Shapley (1962)]

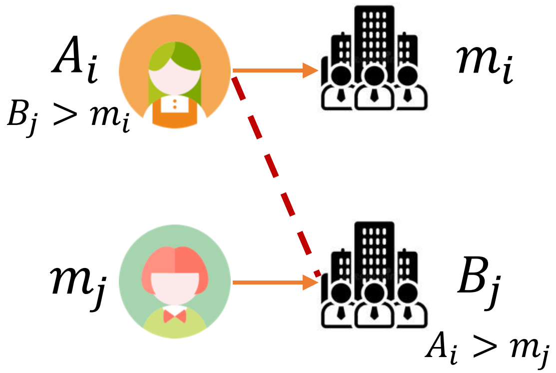
Agents on one side independently propose to agents on the other side according to their preference ranking until no rejection happens

# Gale-Shapley (GS) algorithm: Case 2



# GS properties: Stability

- The GS algorithm returns the stable matching




- Proof sketch
- Suppose there exists blocking pair $(A_i, B_j)$ such that
  - $A_i$ prefers $B_j$ than its current partner $m_i$
  - $B_j$ prefers $A_i$ than its current partner $m_j$
- For $A_i$, it first proposes to $B_j$, but is rejected, then proposes to $m_i$
- This means that $B_j$ must prefers $m_j$ than $A_i$
- Contradiction!

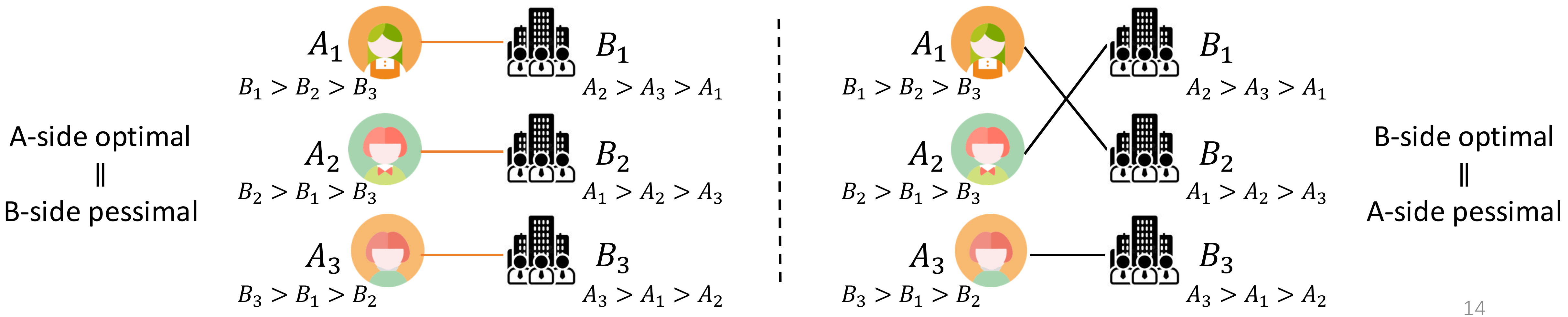
# GS properties: Time complexity

- Each B-side agent can reject each A-side agent at most once
- At least one rejection happens at each step before stop
- $N = \#$ {proposing-side agents}, $K = \#$ {acceptance-side agents}
- $\Longrightarrow$ GS will stop in at most $NK$ steps

The time complexity can be improved as $N^2$ if $N \leq K$ [Kong, Wang and Li, NeurIPS 2024]

# GS properties: Optimality

- Who proposes matters
  - Each proposing-side agent is happiest, matched with the most preferred partner among all stable matchings
  - Each acceptance-side agent is only matched with the least preferred partner among all stable matchings
  - A-side optimal stable matching = B-side pessimal stable matching



# But agents usually have unknown preferences in practice


Can learn them from iterative interactions !

# Bandit learning in matching markets

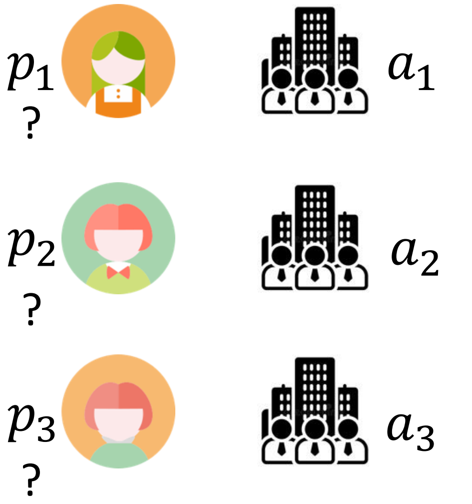
[Liu et al., AISTATS 2020]

Michael Jordan

- $N$ players: $\mathcal{N} = \{p_1, p_2, \dots, p_N\}$
- $K$ arms: $\mathcal{K} = \{a_1, a_2, \dots, a_K\}$
- $N \leq K$ to ensure players can be matched
- $\mu_{i,j} > 0$: (unknown) preference of player $p_i$ towards arm $a_j$
- For each player $p_i$
  - $\{\mu_{i,j}\}_{j \in [K]}$ forms its preference ranking
  - For simplicity, the preference values of any player are distinct
- For each round $t$:
  - Player $p_i$ selects arm $A_i(t)$
  - If $p_i$ is accepted by $A_i(t)$: receive $X_{i,A_i(t)}(t)$ with $$\mathbb{E}\left[X_{i,A_i(t)}(t)\right] = \mu_{i,A_i(t)}$$
  - If $p_i$ is rejected: receive $X_{i,A_i(t)}(t) = 0$

Satisfaction over this matching experience

When would $p_i$ be rejected?




For simplicity, assume arms know their preferences

# Objective

- Minimize the stable regret
  - The player-optimal stable matching
$$\overline{m} = \{(i, \overline{m}_i) : i \in [N]\}$$
  - The player-optimal stable regret of player $p_i$ is
$$\overline{Reg}_i(T) = T\mu_{i,\overline{m}_i} - \mathbb{E}\left[\sum_{t=1}^{T} X_{i,A_i(t)}(t)\right]$$
  - The player-pessimal stable regret $\underline{Reg}_i(T)$
    - Use the objective of the player-pessimal stable matching $\underline{m}$
- Guarantee strategy-proofness
  - Single player can not achieve $O(T)$ reward increase by deviating when others follow the algorithm

# Multi-armed bandits (MAB)

[Lattimore and Szepesvári, 2020]



corresponds to
N=1 player setting

| *Time* | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| *Arm 1* | \$1 | \$0 | | | \$1 | \$1 | \$0 | | | |
| *Arm 2* | | | \$1 | \$0 | | | | | | |

To accumulate as many rewards, which arm would you choose next?

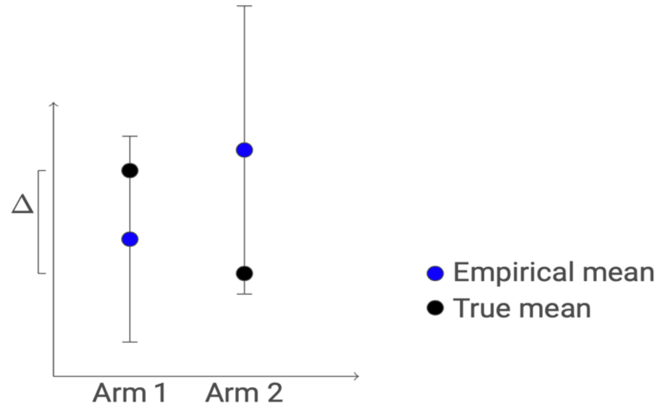
Exploitation V.S. Exploration

# Upper confidence bound (UCB) [Auer et al., 2002]

- With high probability $\geq 1-\delta$ By Hoeffding's inequality

$$\mu_j \in \left[\hat{\mu}_j - \sqrt{\frac{\log 1/\delta}{T_j}}, \hat{\mu}_j + \sqrt{\frac{\log 1/\delta}{T_j}}\right]$$

Sample mean  Number of selections of $a_j$




- Optimism: Believe arms have higher rewards, encourage exploration
  - The UCB value represents the reward estimates
- For each round $t$, select the arm

$$A(t) \in \operatorname{argmax}_{j\in[K]} \left\{\hat{\mu}_j + \sqrt{\frac{\log 1/\delta}{T_j(t)}}\right\}$$

Upper confidence bound (UCB)

Exploitation  Exploration

Without knowing Δ

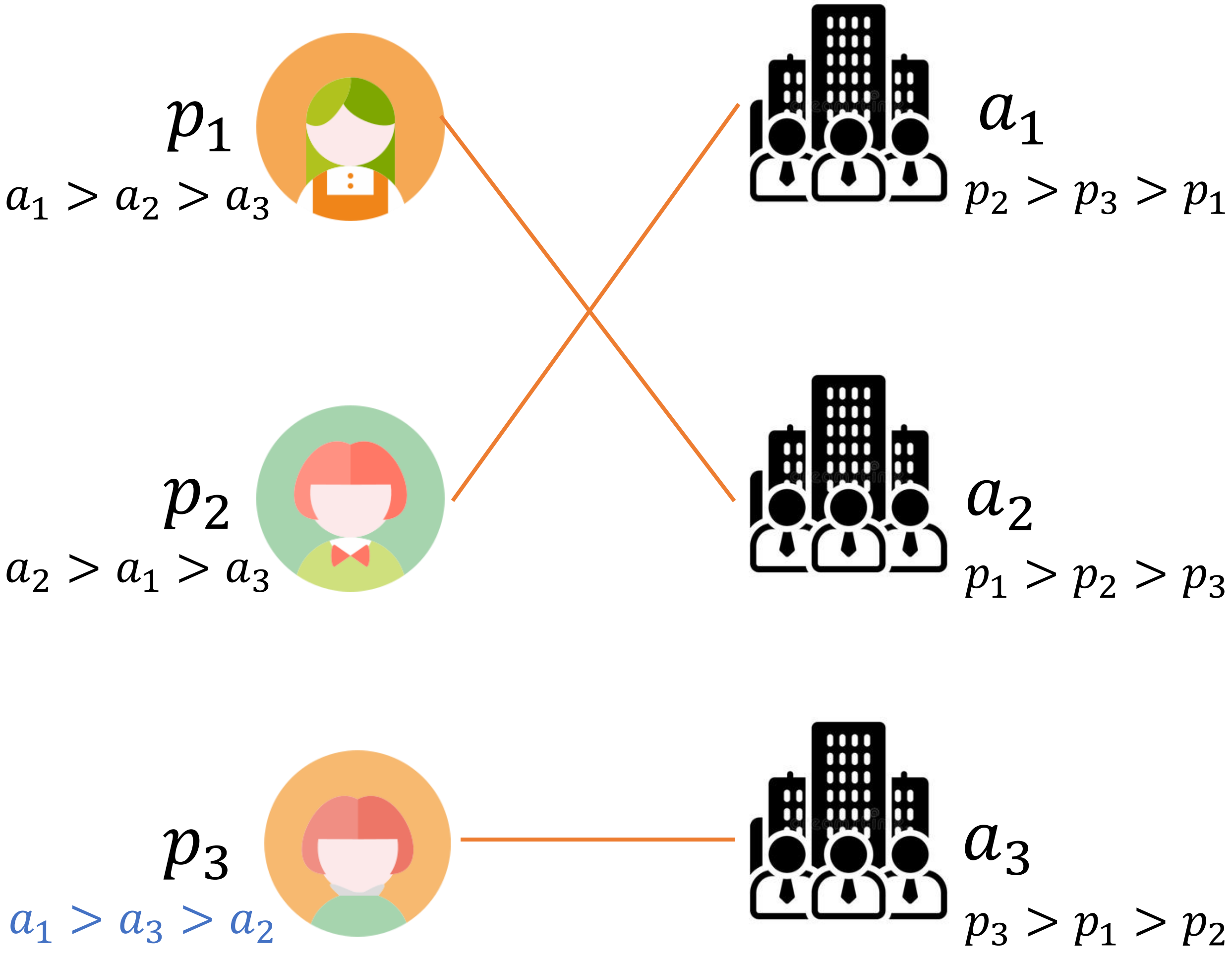
- Regret $O(K \log T/\Delta)$

# Previous works for online matching markets

| | Regret bound | Setting |
|---|---|---|
| Liu *et al.* [2020] | $O\left(K\log T/\Delta^2\right)$<br>$O\left(NK^3\log T/\Delta^2\right)$ | player-optimal, centralized, known $T,\Delta$<br>player-pessimal, centralized |
| Liu *et al.* [2021] | $O\left(\frac{N^5K^2\log^2 T}{\varepsilon^{N^4}\Delta^2}\right)$ | player-pessimal |
| Sankararaman *et al.* [2021] | $O\left(NK\log T/\Delta^2\right)$<br>$\Omega\left(N\log T/\Delta^2\right)$ | unique stable matching |
| Basu *et al.* [2021] | $O\left(K\log^{1+\varepsilon} T + 2^{\left(\frac{1}{\Delta^2}\right)^{\frac{1}{\varepsilon}}}\right)$<br>$O\left(NK\log T/\Delta^2\right)$ | player-optimal<br>unique stable matching |
| Kong *et al.* [2022] | $O\left(\frac{N^5K^2\log^2 T}{\varepsilon^{N^4}\Delta^2}\right)$ | player-pessimal |
| Maheshwari *et al.* [2022] | $O\left(CNK\log T/\Delta^2\right)$ | unique stable matching |

$\Delta$ is the minimum preference gap between different arms among all players, $\varepsilon$ is the hyper-parameter of the algorithm, $C$ is related to the unique stable matching condition and can grow exponentially in $N$

# Why UCB fails to achieve player-optimality?




- When $p_3$ lacks exploration on $a_1$ with $a_1 > a_3 > a_2$ on UCB, GS outputs the matching[1] $(p_1, a_2), (p_2, a_1), (p_3, a_3)$
- $p_3$ fails to observe $a_1$
- UCB vectors do not help on exploration here
- Not consistent with the principle of *optimism in face of uncertainty*

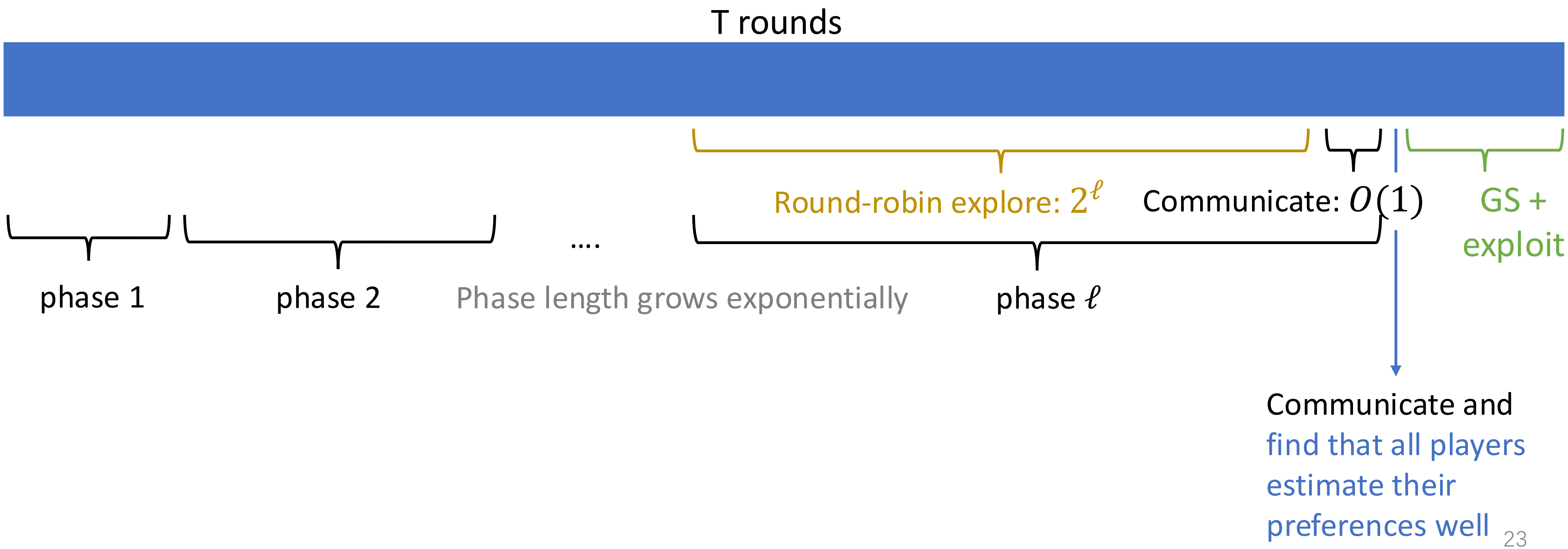
1. When $p_1$ and $p_2$ submit the correct rankings

# How to balance EE in a more appropriate way?

- Exploration-Exploitation trade-off
  - Exploitation goes though with correct rankings by following GS
  - Require enough exploration to estimate the correct rankings
- The UCB ranking does not guarantee enough exploration
- Perhaps design manually?
- To avoid other players' block: Coordinate selections in a round-robin way

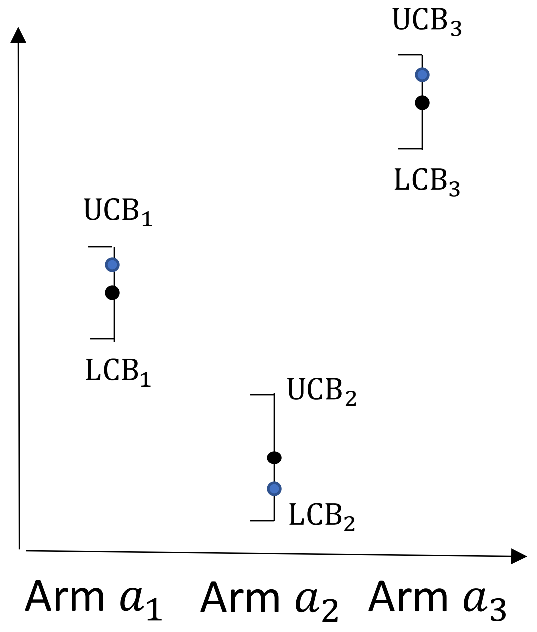
# Explore-then-GS (ETGS) [Kong and Li, SODA 2023]

- Avoid unnecessary exploitation before estimating preferences well
  - Only when all players estimate well, enter GS + exploit



# ETGS implementation: Communication

- At communication block: players determine whether all players estimate their preference rankings well
- For $p_i$
  - If there exists a ranking $\rho_i$ over arms such that
    - The confidence intervals of all arms are disjoint
  - Note: this estimated ranking is accurate w.h.p.
- How to communicate with others?



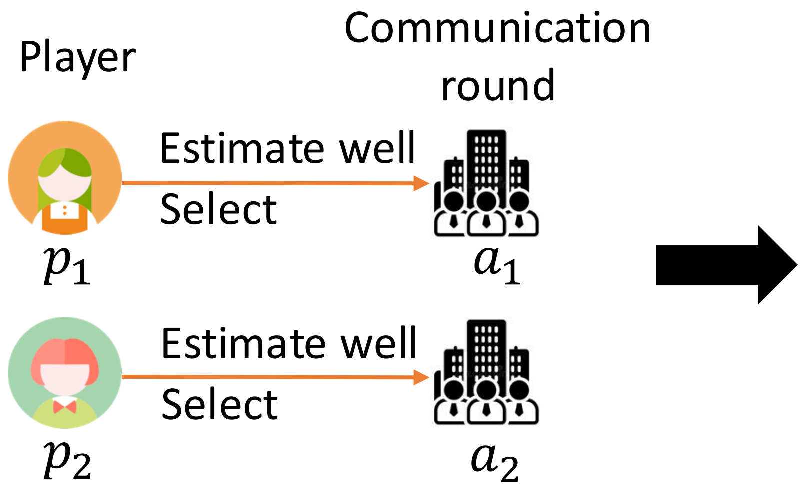

player $p_i's$ preference values



Remark: each player identifying the arms ranked in the first N+1 is enough to find the player-optimal stable matching.

# ETGS implementation: Communication (cont.)

- Based on observed all players' matching outcomes [KL, 2023]
  - If $p_i$ has estimated well with ranking $\rho_i$: select arm $a_i$
  - Else: Select nothing

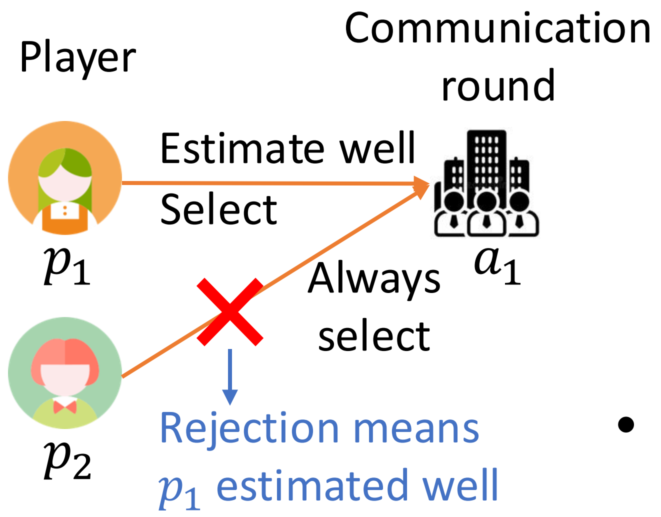



At the communication round, if $p_i$ observes that all players have been matched:

Then all players estimate their preference well

# ETGS implementation: Communication (cont.)

- Based on players' own matching outcomes [Zhang et al., 2022]
  - Communicate based on every pair of players
    - $p_i$ can transmit information $\{0,1\}$ to $p_{i'}$ based on $a_j$ $(p_i > p_{i'})$
    - In the corresponding round, $p_{i'}$ always selects $a_j$
    - If $p_i$ finished exploration, selects $a_j$
      - $p_{i'}$ is rejected, receives information 1
    - Otherwise, $p_i$ do not select $a_j$
      - $p_{i'}$ is accepted, receive information 0
  - If a player cannot receive others' information (all arms prefer this player than others)
    - The player can directly exploit the stable arm
    - Others cannot block it



# ETGS: Regret analysis [Kong and Li, SODA 2023]

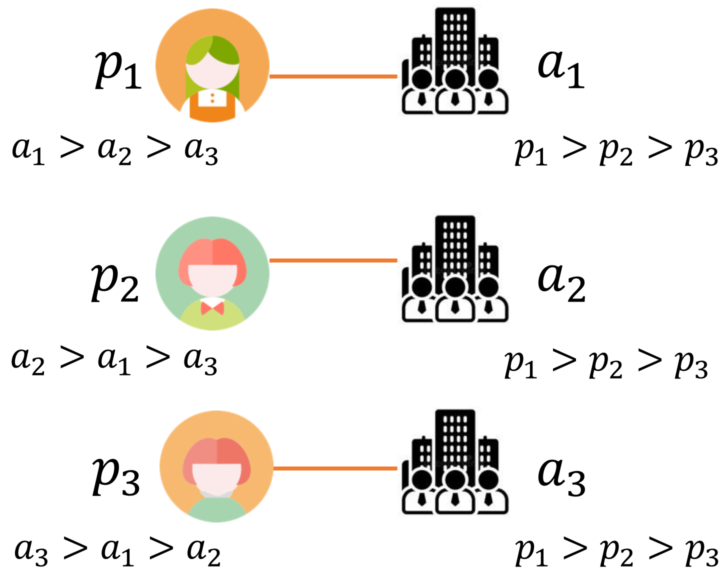
- Exploration is enough $\Longrightarrow$ Estimated ranking is correct $\Longrightarrow$ All players enter the GS + exploit phase and find the player-optimal stable matching
- The player-optimal regret comes from exploration and communication

$$\overline{Reg}_i(T) = O\left(\frac{K\log T}{\Delta^2} + \log\left(\frac{K\log T}{\Delta^2}\right)\right)$$

- What is the optimal regret that an algorithm can achieve?

Remark: $\Delta$ can be improved as the minimum gap between the first N+1 ranked arms among all players.

# Lower bound [Sankararaman et al., AISTATS 2021]




- Optimally stable bandits
  - All arms have the same preferences
  - $\Longrightarrow$ Unique stable matching exists
  - The stable arm of each player is its optimal arm
- For any player $p_i$
  - Its stable arm is $a_i$
  - $a_i$ prefers $p_1, p_2 \dots \dots p_{i-1}$ than $p_i$
  - $T_{i,j}$: the number of times that $p_i$ selects $a_j$

The minimum regret that $p_i$ may suffer at any round

$$\overline{Reg}_i(T) \geq \max\left\{\Delta_{i,i,j} \sum_{j \neq i} T_{i,j}\,, \quad \Delta_{i,\min} \sum_{i' < i} T_{i',i}\right\}$$

$p_i$ selects sub-optimal arm $a_j$

The optimal arm $a_i$ is occupied by a higher-priority player

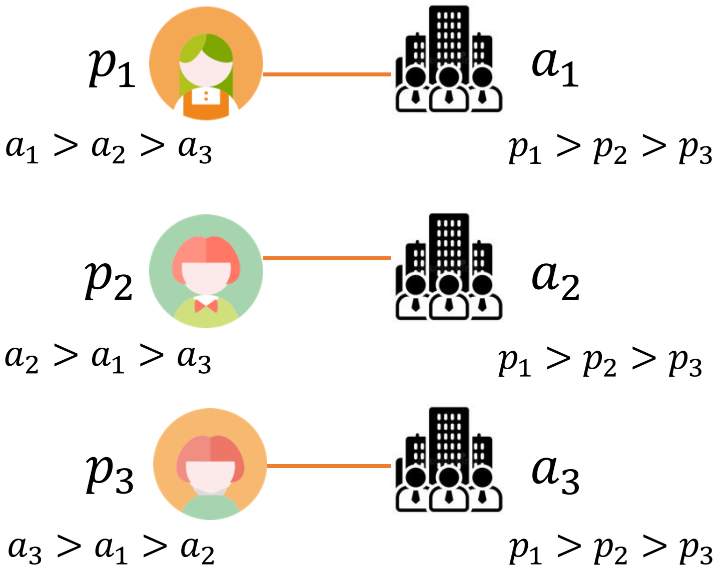
# Lower bound (cont.)

- How many times does $p_i$ select a sub-optimal arm $a_j$ ?
  - To distinguish the sub-optimal arm $a_j$ from the optimal arm $a_i$
  - $p_i$ needs to observe this arm

$$\Omega\left(\frac{\log T}{\Delta_{i,i,j}^2}\right) \text{ times}$$

- $K$ sub-optimal arms cause regret

$$\Omega\left(\sum_{j\neq i} \frac{\log T}{\Delta_{i,i,j}^2} \cdot \Delta_{i,i,j}\right) = \Omega\left(\frac{K\log T}{\Delta}\right)$$



# Lower bound (cont.)

- How many times does $a_i$ is occupied by a higher-priority player $p_{i'}$?
  - To distinguish the sub-optimal arm $a_i$ from the optimal arm $a_{i'}$
  - $p_{i'}$ needs to observe this arm

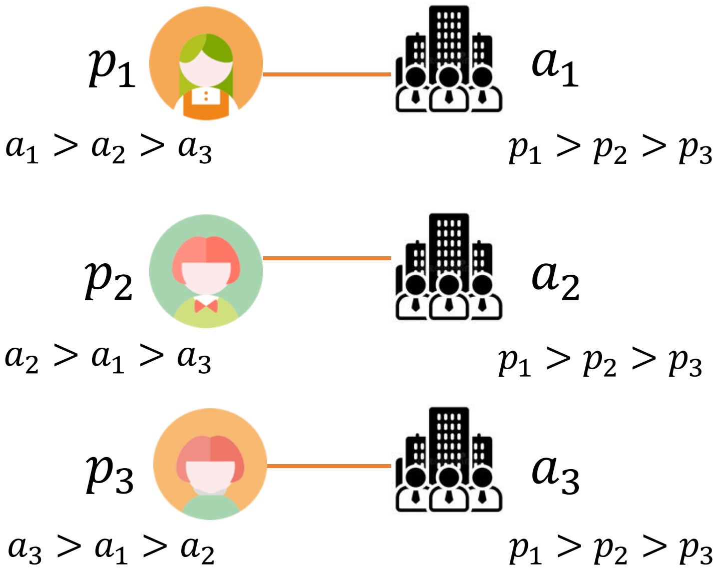
$$\Omega\left(\frac{\log T}{\Delta^2_{i',i',i}}\right) \text{ times}$$

- $N$ higher-priority players cause regret

$$\Omega\left(\sum_{i'<i} \frac{\log T}{\Delta^2_{i',i',i}} \cdot \Delta_{i,\min}\right) = \Omega\left(\frac{N\log T}{\Delta^2}\right)$$

- The stable regret satisfies

$$\overline{Reg}_i(T) \geq \Omega\left(\frac{N\log T}{\Delta^2} + \frac{K\log T}{\Delta}\right)$$

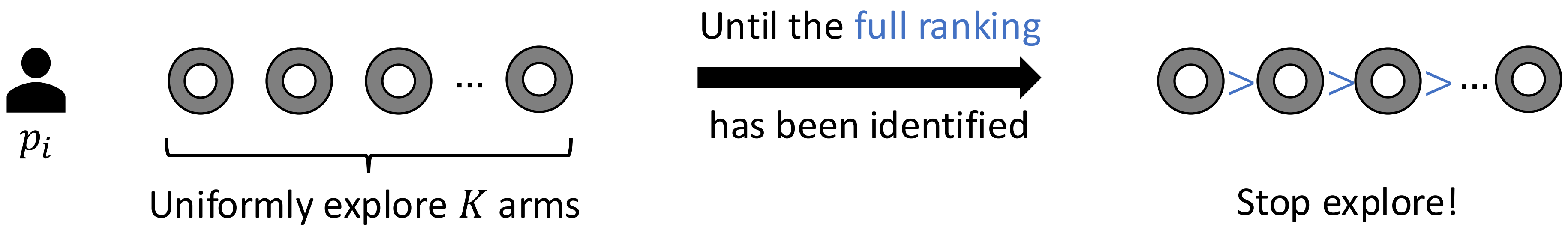





Remark: Δ can be improved as the minimum gap between the player-optimal stable arm and the next preferred one among all players.

# Sub-optimality of ETGS

- Needs to identify the full ranking among $K$ arms

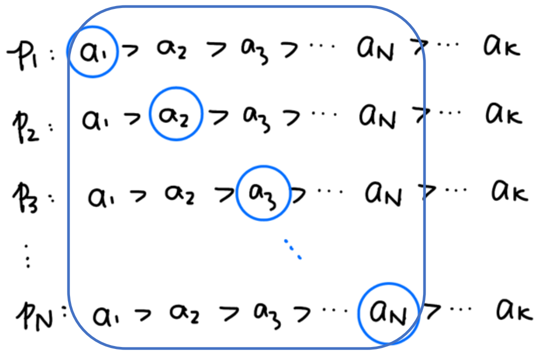


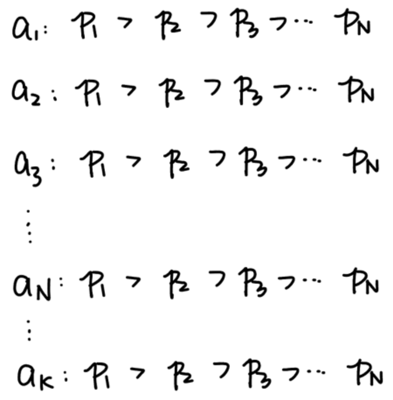

- But for the Gale-Shapley algorithm, what is the real complexity to find the optimal stable matching?
  - Whether it is necessary to determine the full ranking over $K$ arms

# Key observation of GS properties

Could the dependence of $K$ be improved as $N$?

- The optimal stable arm must be the first N-ranked
  - The player moves to the next arm only if this arm is occupied by another player
  - N players at most occupy N arms



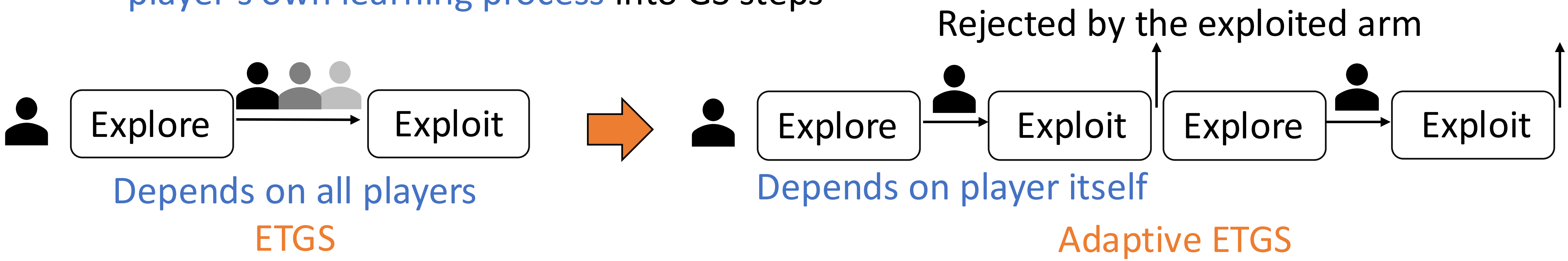

- The GS algorithm proceeds for at most $N^2$ steps
  - Those N arms can reject each of N players for at most once

# Strategic behavior of ETGS


- If ∃ a player whose stable arm is the least preferred one
- He can always report that he has not finished exploration
- All players fail to enter the exploitation phase
- This player: Always match better arms during exploration, $O(T)$ reward increase
- Other players: $O(T/K)$ times match worse arms, $O(T)$ reward decrease
- Not strategy-proof!

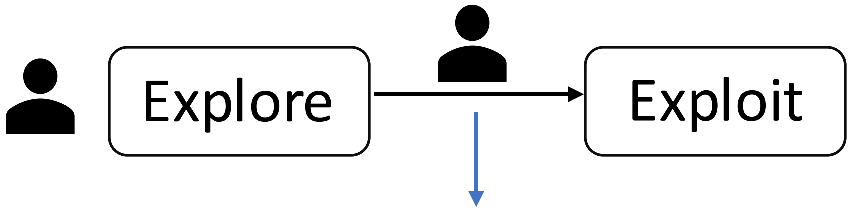
# Improvement: Adaptive ETGS [KL, AAAI 2024]

- Idea: Instead of starting GS + exploitation with all players' agreement, integrating each player's own learning process into GS steps




- Players cooperatively explore arms in a round-robin manner
- Once a player identifies the most preferred one, starts exploiting this arm
- If the exploited arm is occupied by a higher-priority player (the arm "rejects" the player)
  - Explore the next most preferred arm (enter the next step of GS)

# Adaptive ETGS: Strategic behavior




Have identified the optimal arm. What to report?

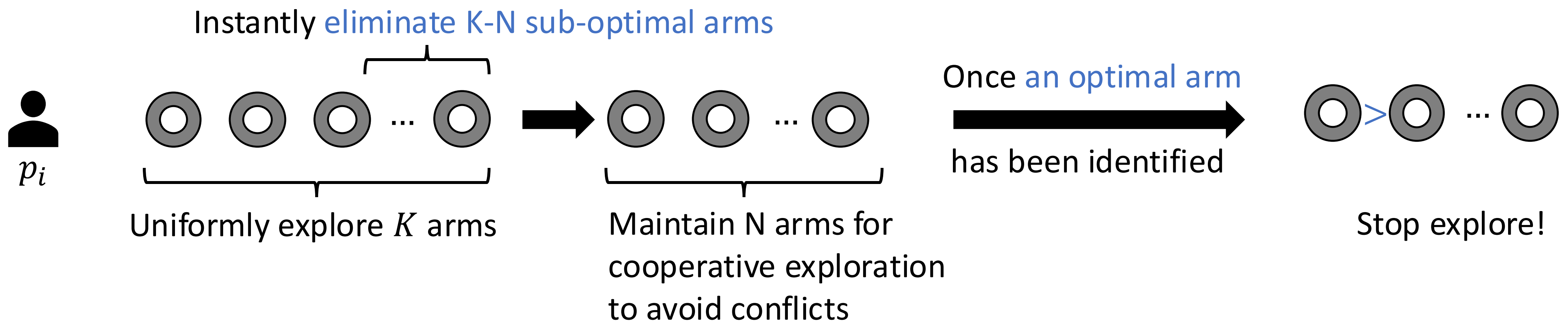
**How about reporting NOT?**
- Equivalent to delayed entering GS in the offline setting
- Cannot change the final matching results

**How about reporting a non-optimal arm?**
- Equivalent to misreporting rankings in the offline GS
- Cannot improve the final matched partner

- Is strategy-proof: Single player can not obtain $O(T)$ reward increase (improve the final matched arm) by misreporting the exploration status

# Adaptive ETGS: Regret [KWL, NeurIPS 2024]

- Arrangement of the exploration process




- The player-optimal stable regret of each player $p_i$ satisfies

$$\overline{Reg}_i(T) \leq O\left(\frac{N^2 \log T}{\Delta^2} + \frac{K \log T}{\Delta}\right)$$

$\Delta$ is the minimum preference gap between any arms that have higher ranking than the arm after the player-optimal stable one.

| Regret type | Regret Bound | Communication type | Strategy-proofness | References |
|---|---|---|---|---|
| Player-optimal | $O\left(\frac{K\log T}{\Delta^2}\right)$ | Centralized, known $\Delta$ | ? | [Liu et al., AISTATS 2020] |
| Player-pessimal | $O\left(\frac{NK\log T}{\Delta^2}\right)$ | Centralized | ? | |
| | $O\left(\frac{N^5K^2\log^2 T}{\rho^{N^4}\Delta^2}\right)$ | Decentralized, observed matching outcomes | No | [Liu et al., JMLR 2021]<br>[KYL, IJCAI 2022] |
| Unique | $O\left(\frac{NK\log T}{\Delta^2}\right)$ | Decentralized | ? | [Sankararaman et al., AISTATS 2021; Basu et al., ICML 2021; Maheshwari et al., NeurIPS 2022] |
| | $O\left(\frac{N\log T}{\Delta^2}\right)$ | Centralized | ? | [Wang and Li, TCS 2024; KWL, NeurIPS 2024] |
| Optimal stable (Unique) | $\Omega\left(\frac{N\log T}{\Delta^2}+\frac{K\log T}{\Delta}\right)$ | Decentralized | / | [Sankararaman et al., AISTATS 2021] |
| Player-optimal | $O\left(K\log^{1+\varepsilon} T+2^{\left(\frac{1}{\Delta^2}\right)^{1/\varepsilon}}\right)$ | Decentralized | ? | [Basu et al., ICML 2021] |
| | $O\left(\frac{K\log T}{\Delta^2}\right)$ | Decentralized, observed matching outcomes | No | [Kong and Li, SODA 2023] |
| | | Decentralized | No | [Zhang et al., NeurIPS 2022] |
| | $O\left(\frac{N^2\log T}{\Delta^2}+\frac{K\log T}{\Delta}\right)$ | Decentralized | Yes | [KWL., NeurIPS 2024] |
| Indifference stable | $O\left(\frac{NK\log T}{\Delta^2}\right)$ | Decentralized | ? | [KTLLLL, ICLR 2025] |

# Other setting variants

- Many-to-one matching markets
- Strategic behaviors
- Contextual information and indifferent preferences
- Non-stationary preferences
- Two-sided/multi-sided unknown preferences
- Markov matching markets
- Multi-sided matching markets

# Many-to-one markets: Results overview

| Setting | Regret type | Regret Bound | Communication type | Strategy-proofness | References |
|---|---|---|---|---|---|
| Responsiveness | Player-optimal | $O\left(\frac{K\log T}{\Delta^2}\right)$ | Centralized, known $\Delta$ | ? | [WGYL, CIKM 2022] |
| | Player-pessimal | $O\left(\frac{NK^3\log T}{\Delta^2}\right)$ | Centralized | ? | |
| | | $O\left(\frac{N^5K^2\log^2 T}{\kappa^{N^4}\Delta^2}\right)$ | Decentralized, observed matching outcomes | No | |
| | Player-optimal | $O\left(\frac{K\log T}{\Delta^2}\right)$ | Decentralized, observed matching outcomes, $N \le K \cdot \min_j C_j$ | No | [Kong and Li, AAAI 2024] |
| | | $O\left(\frac{N\min\{N,K\}C\log T}{\Delta^2}\right)$ | Decentralized, observed matching outcomes | Yes | |
| | | $O\left(\frac{\max\{N,K\}\log T}{\Delta^2}\right)$ | Decentralized | No | [Zhang and Fang, AAMAS 2024] |
| Substitutability | Player-pessimal | $O\left(\frac{NK\log T}{\Delta^2}\right)$ | Decentralized, observed matching outcomes, known arms' preferences | ? | [Kong and Li, AAAI 2024] |

# Open problems

- What is the optimal regret order?
  - $\Theta(N \log T / \Delta^2)$?
- How to guarantee strategy-proofness when players have more freedom?
  - The player needs to determine not only which 'optimal arm' to report
- How to generalize the setting and what is the optimal regret in these settings?
  - How to deal with players' indifferent preferences?
  - How to utilize the contextual information to accelerate the learning efficiency?
  - How to handle asynchronous agents?
- Maximum matching VS. Stable matching?

Thanks!
&
Questions?

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